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# THE ALGEBRAIC GEOMETRY OF THE GLUON CONDENSATE

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**Abstract**

In this note an irreducible representation (1) of the center D of the Dirac ring with the commuting operators $E_{23}, E_{14}, E_{05}$ of spin, parity and isospin is related to a fundamental tetrahedron whose 6 edges are ' fix-lines' labeled by the $\pm i$ eigenvalues of the $E_{\mu\nu}$. This tetrahedron is invariant under the tetrahedral group T and the associated algebraic variety is the Cayley cubic of equation (10)and Figs. 2 and 3.There are 9 characteristic lines and 3 tangent cones at a, b, and d . The surface of each tangent cone is believed to the home of quarks with spin $\frac{1}{2}$ and this contribution shows how the properties of the fix-lines determine the charges carried by the quarks as well as the gluon propagator at c of Fig.3 which is not a tangent cone. The surface of Fig.2 is ruled by fibers that are gluon paths with changes of color charge where the paths intersect the conic surfaces. However, because T is not invariant under time reversals, quarks are short-lived but there is a quartic variety that is invariant under reflections which yields the nuclear quadrupole surface normally accounted for by the strong interaction which now becomes a purely geometric requirement according to the Lemma after equation (7).

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## 1 Introduction

This note is an appendix to Ref. [4] on The Algebraic Geometry of the Quark that demonstrated the connection between the Cayley cubic and nuclear structure based on equation (1) which is an irreducible representation or minimum left ideal of the center D of the Dirac ring [3]. The fundamental association was made by Barth and Nieto [1] who showed that the rotations $E_{23}$ in 3-space, $E_{05}$ in isospace and $E_{14}$ in 4-space are a set of one of the 15 synthemes in the projective space $P^3$. They specify a pair of ' fix-lines' lying in the $\pm i$ eigenspaces determined by the relation $E_{\mu\nu}{}^2 = -1$. The 6 fix-lines are the edges of a fundamental tetrahedron that may be inscribed in a cube as illustrated by Fig. 1.Their identification with $E_{23}, E_{14}.E_{05}$ will be presented below after we have examined the Cayley cubic in more detail.

In a seminal paper Slansky [10] used the exceptional Lie group $E_6$ to find the quark and gluon charges and color representations which are set out in Table 21. There are 27 weights in $E_6$ that correspond to the 27 lines on a cubic. However the tetrahedron is invariant under the tetrahedral group. But in this case the cubic surface is singular and there remain only 9 lines on a Cayley cubic ( cf.[7] ,Section 4.1.3) .Equation (9) defines this cubic and the algebraic picture is illustrated in Figs. 2 and 3. where the 9 lines are aa, aa',aa"; bf, de: dd', bb'; cc'and ef. [8]. In particular the lines aa',bf and aa", de are on conical surfaces deemed to be the home of quarks with spin $\frac{1}{2}$ and fractional charge whose properties spring from the fix-lines ab and ad.

Fig.1 is Fig.2 rotated through $\pi$ about aa,cc' so that the vertex c appears and is shown by Fig.3 to be a gluon propagator. We let the fix-lines ab,ad correspond to the operators $E_{05}, E_{50}$ with isospin $T_3 = \pm 1/2$ , in which case the quark charge along de will be $Q = T_3 + 1/6 = 2/3$ . This is also the charge along aa" . That along bf would be Q = - 1/2 + 1/6 = -1/3 so we find a proton udu. If on the other hand we consider the quark charge along aa' where $T_{\ 3} = -1/2$ we find a neutron. Color charges will be introduced in the next section. The remaining fix-line on the plane abd is bd associated with $E_{\ 14}$ which will be seen in the next section to be a parity operator. But a change from a left-handed to a right-handed coordinate system is accompanied by charge conjugation from $T_{\ 3} = -1/2$ at b to $T_{\ 3} = +1/2$ at d, by the CP-theorem. A reflection of the coordinate system would simply interchange b and d and the roles of proton and neutron.

Referring to Fig.3 which is Fig.2 rotated by $\pi/4$ we see that c is a gluon propagator so we come to the remaining 3 fix-lines associated with it. The spins $E_{\ 23}, E_{\ 32}$ with the eigenvalues $\pm 1/2$ are carried by cb ,cd so that there is no spin at c and no projections onto cones of the cubic. However the last fix-line $E_{\ 41}$ associated with ca has projections aa,cc' and compliments $E_{\ 14}$ in that a reflection $a' \leftrightarrow a''$ would also interchange $p \leftrightarrow n$. The last of the 9 lines on the cubic is ef which defines a boundary.

## 2 The Gluon Condensate

The central equation is the irreducible representation or one-form

$$\frac{1}{4}\Psi = (iE_4\psi_1 + E_{23}\psi_2 + E_{14}\psi_3 + E_{05}\psi_4)e \qquad (1)$$

of a minimum left ideal of the center D of the Dirac ring.Here we use Eddington's transparent notation and associate the commuting operators $E_{23}$,$E_{14}$ and $E_{05}$respectively with independent rotations in 3-space,4-space and isospace which correspond to the spin $\sigma$,$\pi$ and charge $T_3$ carried by a single nucleon.$E_4$ is the unit $4 \times 4$ matrix while $\psi_2$,$\psi_3$ and $\psi_4$ are half angles of rotation or 'Turns' [2]; e is a primitive idempotent.To see how $E_{14}$ is related to parity we notice that a rotation through $\pi$ about $t$ will will send $x$ to $-x$ without inverting the time axis but instead changing to a left-handed coordinate system. Eddington's $E$ numbers are mapped into the Dirac $\gamma$ matrices by

$$\gamma_\nu = iE_{0\nu}, E_{\mu\nu} = E_{\rho\mu}E_{\rho\nu} = -E_{\nu\mu}, E^2_{\mu\nu} = -1$$
$$E_{\mu\nu}E_{\sigma\tau} = E_{\sigma\tau}E_{\mu\nu} = iE_{\lambda\rho} < \nu = 1, \ldots, 5 \qquad (2)$$

The many nucleon representation is found by calculating the tensor product of (1) with itself when we encounter the de Broglie algebras with the $4^A \times 4^A$ basis elements

$$E_{\mu\nu} = E_4 \otimes \cdots \otimes E_4 \otimes E_{\mu\nu} \otimes E_4 \otimes \cdots \otimes E_4 \qquad (3)$$

such that $E_{\mu\nu}$ is in the l-th position and $[E^l_{\mu\nu}, E^{(l+1)}_{\mu\nu}] = 0$. Then the IR's of the enveloping algebra $A(\gamma)$ are the 3-form [3,5]

$$\psi^{(A)} = \sum_\lambda C_{[\lambda]}P_{[\lambda]} \qquad (4)$$

where P is a projection operator

$$P_{[\lambda]} = i^{-A}(i_A{\psi_1}^{\lambda_1}\psi_2^{\lambda_2}\psi_3^{\lambda_3}\psi_4^{\lambda_4} + \eta^{(A)}_{23}\psi_1^{\lambda_2}\psi_2^{\lambda_1}\psi_3^{\lambda_4}\psi_4^{\lambda_3}$$
$$+ \eta^{(A)}_{14}\psi_1^{\lambda_3}\psi_2^{\lambda_4}\psi_3^{\lambda_1}\psi_4^{\lambda_2} + \eta^{(A)}_{05}\psi_1^{\lambda_4}\psi_2^{\lambda_3}\psi_3^{\lambda_2}\psi_1^{\lambda_1})\epsilon_A \qquad (5)$$

satisfying

$$P^2_{[\lambda]} = P_{[\lambda]}\psi_1^{\lambda_1}\psi_2^{\lambda_2}\psi_3^{\lambda_3}\psi_4^{\lambda_4} \tag{6}$$

and $[\lambda] = [\lambda_1\lambda_2\lambda_3\lambda_4]$ labels a state.
Here $\epsilon_A = \epsilon \otimes \cdots \otimes \epsilon$ is idempotent so that (5) has precisely the same form in configuration space $A = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ as (1) has in a 4-space. The essential difference lies in the fact that the operators $E_{23}, E_{14}, E_{05}$ are replaced by their tensor products

$$\eta_\nu^{(A)} = E_{0\nu} \otimes \cdots \otimes E_{0\nu} = E^1_{0\nu}E^2_{0\nu}\dots E^A_{0\nu}, \eta^{(A)}_{\mu\nu} = \eta^{(A)}_\mu \eta^{(A)}_\nu = E^1_{\mu\nu}E^2_{\mu\nu}\dots E^A_{\mu\nu} \tag{7}$$

Because according to (3) the matrices (7) are diagonal in the $E_{\mu\nu}$ we can consider $E^l_{23}, E^l_{14}, E^l_{05}$ to define 6 fix-lines on a tetrahedron. This is the same fundamental tetrahedron because the algebraic variety that preserves it under the Dirac operators of rotation and reflection is a quartic which yields the nuclear quadrupole shape that is found experimentally to be independent of the number of nucleons [5]. Normally this is accounted for by positing a 'strong force' which we can now see is a purely geometric requirement that makes no appeal whatsoever to mechanics and we have the

Lemma: The strong interaction is a property of a Lorentz invariant particle-like representation in a symplectic manifold of 4-space.

The traces of (7) are twice the de Broglie operators

$$(E^1_{\mu\nu} + E^2_{\mu\nu} + \cdots + E^A_{\mu\nu}) \tag{8}$$

which add the spins that label the state $[\lambda]$ where it has been shown in [5] that a canonical labelling is obtained by postulating that $(\lambda_2 + \lambda_3)$ is the number of nucleons with negative spin and $(\lambda_3 + \lambda_4)$ is the number of protons Z.

On the other hand we have already seen that the tetrahedral group T is not invariant under reflections $E_{14}$. In particular the Cayley cubic will not be invariant under time reversals if there is no space reflection changing an up to a down quark which is the reason why quarks are short-lived and Fig.2 only appears in high energy experiments as jets of quarks and gluons on the cones.

However we can now depict any high energy nucleus by the Cayley cubic

$$u_0u_1u_2 + (u_0u_1 + u_0u_2 + u_1u_2)u_3 = 0 \tag{9}$$

which may be written

$$-5(xy^2 + yx^2 + xz^2 + zx^2 + zy^2 + yz^2 + xyz) + 2(xy + yz + xz) = 0 \quad (10)$$

in tetrahedral coordinates

$$x = u_{\ 0}/v, y = u_{\ 1}/v, z = u_{\ 2}/v \quad (11)$$

together with a projective plane at infinity

$$\frac{1}{2}(2/5 - x - y - z) = u_3/v \quad (12)$$

Finally the ruled surfaces of Figs.2,3 may be considered to be the paths of many gluons which according to current theory bind the quarks. One path is outlined in Fig.2 that is algebraically a fibre ([6],Section v,2). This fibre meets a quark surface at point b and if it is, say, red could lead to the exchange $r \rightarrow Ar \rightarrow b$ (where A stands for 'Anti-' and b is 'blue').But because there can be several quarks on the conical surface the gluon leaving this surface could now carry another quark's blue colour charge to the next vertex a responsible for the exchange $b \rightarrow Ab \rightarrow g$. The gluon carrying the green colour charge to d would then suffer the exchange $g \rightarrow Ag \rightarrow r$ so that another red colour charge returns to the gluon propagator c shown in Fig.3 and so on.There is rotational symmetry about an axis passing through c in Fig.2 (oriented so that c is in the centre of abd).

# 3 Conclusion

Since the tetrahedral group T is not invariant under time reversals Fig.2 is not relevant for the study of stable nuclear structures which are rotating quadrupoles and even octupoles and hexadecupoles that would be indistinguishable from a deformed spherical shell [5]. For example the strong interactions follow from the fact that all of the nuclear fix-lines lie on the same fundamental tetrahedron that underlies the Kummer quartic. Finally since the tetrahedron of figure 1 gives rise to surface physics described by figure 2 and 3, we could be looking at a holographic version of conventional three-dimensional quantum nuclear physics and strings in the interior of the surfaces [11]

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# References

[1] W.Barth and I.Nieto, Abelian surfaces of type (1,3) and quartic surfaces with 16 skew lines,*J.Algebraic Geometry,***3**(1994),173-222.

[2] L.C.Biedeharn and J.Louck, Angular Momentum in Quantum Physics, *Encyclopaedia of Mathematics and its Applications, Vol.8,*Addison Wesley,Reading,Mass.1981.

[3] J.A. de Wet, A group theoretic approach to the many nucleon problem,*Proc.Camb.Philos.Soc.,***70**(1971) 485-496.

[4] J.A. de Wet, Algebraic geometry of the quark,*J. of Dynamical Systems & Geometric Theories,***3**(2005)77-85.

[5] J.A. de Wet, Nucleon deuteron structure and nucleon entanglement, *Intrn. Math. Journal,* **5** (2004), 49-60.

[6] R. Hartshorne, Algebraic Geometry, *Graduate Texts in Mathematics 52,* Springer, 1977.

[7] Bruce Hunt, The Geometry of some special Arithmetic Quotients, *Lecture Notes in Mathematics 1637,* Springer, 1996.

[8] Oliver Labs,*Algebraic Geometry Group,*Univ. of Mainz (2005)

[9] R.Morris, *http://www.singsurf.org/singsurf/SingSurf.html,*(2005)

[10] R.Slansky, Group theory for unified model building, *Phys.Reports***79**, Reprinted in *Unity of Forces in the Universe ,Vol.1 ,Ed.A.Zee,*World Scientific(1982).

[11] E. Witten, Ante de sitter, *Space and Holography,* Adv Theor. Math, Phys. 2 (1998) 253 - 291

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Fig.1 Fundamental Tetrahedron inscribed in a Cube.

Fig.2 Quarks on a Cayley Cubic Surface

Fig.3 Fig.2 Rotated through *pi* about the Vertical Axis.